On the Equilibrium Strategies for Time-Inconsistent Problems in Continuous Time

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Outline

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  - **Equilibrium strategies** if the agent is aware of the inconsistency and unable to commit her future selves
- Equilibrium strategies are rational choice of an agent with no self control, who thus considers her selves at different times to be different players in a **sequential game**.

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  - where x stands for the Markovian state at that time; and
  - $\mathbf{u}$ represents the agent's strategy.
- The agent considers **Markovian strategies**, so $\mathbf{u}$ is a function of time $s \in \{0, 1, \ldots, T-1\}$ and the Markovian state at that time.

Equilibrium Strategies in Discrete-Time Decision Problems (Cont'd)

- A strategy $\hat{\mathbf{u}}$ is an **equilibrium strategy** if at any time $t \in \{0, 1, \ldots, T-1\}$ and in any state x at that time, any deviation of the agent's self at time t from $\hat{\mathbf{u}}(t, x)$, given that her future selves still follow $\hat{\mathbf{u}}$, is suboptimal, i.e.,

$$J(t, x; \mathbf{u}_{t,u}) \le J(t, x; \hat{\mathbf{u}})$$

for any possible action u the agent's self at time t can take, where

$$\mathbf{u}_{t,u}(s, y) := \begin{cases} u, & (s, y) = (t, x), \\ \hat{\mathbf{u}}(s, y), & s \neq t. \end{cases}$$

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- The above definition follows the standard definition of equilibria in the game theory, i.e., **subgame perfect equilibrium**

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- To overcome the above difficulty, Strotz (1955-1956), Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), and Björk and Murgoci (2010) propose that the agent's self at each time t can implement her strategy in an infinitesimally small, but positive, time period, e.g., $[t, t+\epsilon)$.

Equilibrium Strategies in Continuous-Time Decision Problems (Cont'd)

- Formally, they define

$$\mathbf{u}_{t,\epsilon,\mathbf{a}}(s,y) := \begin{cases} \mathbf{a}(s,y), & s \in [t, t+\epsilon), \\ \hat{\mathbf{u}}(s,y), & s \notin [t, t+\epsilon), \end{cases}$$

where $\mathbf{a}$ stands for the strategy that the agent's self at time t chooses to implement in the period $[t, t+\epsilon)$.

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  where $\mathbf{a}$ stands for the strategy that the agent's self at time t chooses to implement in the period $[t, t+\epsilon)$.

- They define $\hat{\mathbf{u}}$ to be an equilibrium strategy if the following holds for any time $t \in [0, T)$, Markovian state x, and action $\mathbf{a}$:

$$\liminf_{\epsilon \downarrow 0} \frac{J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) - J(t, x; \hat{\mathbf{u}})}{\epsilon} \leq 0.$$

A Problem

- As first noted by Björk et al. (2017), the above condition, which is a **first-order condition**, does not necessarily imply

$$J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) \leq J(t, x; \hat{\mathbf{u}})$$

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  for sufficiently small ϵ; the latter condition is a natural definition of equilibrium strategies from the game-theoretic point of view and also consistent with its discrete-time counterpart.
- The literature, however, still use the notion of equilibrium strategies based on the first-order condition, which we refer to as **weak** equilibrium strategies in the following.

Other Issues Overlooked in the Literature

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- First, in the discrete-time setting, the strategy u that the agent's self at time t can implement is constant by the nature of the setting, but in the continuous-time setting, the agent can implement her strategy $\mathbf{a}$ in a positive time period $[t, t+\epsilon)$ and thus $\mathbf{a}$ can be non-constant.

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- Second, in the discrete-time setting, when examining whether the agent would deviate from a given strategy $\hat{\mathbf{u}}$, one only needs to consider *alternative* strategies, i.e., those u such that $u \neq \hat{\mathbf{u}}(t,x)$.
- In the continuous-time setting, however, even if $\mathbf{a}(t,x) = \hat{\mathbf{u}}(t,x)$, $\mathbf{a}(s,y)$ can be different from $\hat{\mathbf{u}}(s,y)$ for $s \in (t, t+\epsilon)$ in which the agent's self at time t implements her strategy.

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- In one of them, which is referred to as **regular** equilibrium strategies, at each time t and in each Markovian state x the agent's self at that time is not allowed to take a strategy $\mathbf{a}$ with $\mathbf{a}(t, x) = \hat{\mathbf{u}}(t, x)$.

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- In both notions, we specify the set of strategies the agent's self at each time can take, e.g., the set of all constant strategies.

Contributions (Cont'd)

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- We further provide an example of optimal consumption to show that a weak equilibrium strategy may not be a regular equilibrium.

Literature

- Examples of works on the study of equilibrium strategies in discrete-time time-inconsistent decision problems: Peleg and Yaari (1973), Laibson (1997), O'Donoghue and Rabin (1999), Barberis (2012), Harris and Laibson (2001), Krusell et al. (2002), Krusell and Smith (2003), and Sorger (2004), Björk and Murgoci (2014)

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- Examples of works on the study of weak equilibrium strategies in continuous-time time-inconsistent decision problems: Strotz (1955-1956), Pollak (1968), Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), Björk and Murgoci (2010), Yong (2012), Ebert et al. (2017), Björk et al. (2017)

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  - They prove the existence and a characterization of the equilibrium strategies.
- Their framework, however, cannot be applied to most time-inconsistent problems in the literature.

Outline

Stochastic Control Problem

- Consider an agent who makes dynamic decisions in a given period $[0, T]$, and for any $t \in [0, T)$, the agent at that time faces the following stochastic control problem:

$$\begin{cases} \max_{\mathbf{u}} & J(t, x; \mathbf{u}) \\ \text{subject to} & dX^{\mathbf{u}}(s) = \mu(s, X^{\mathbf{u}}(s), \mathbf{u}(s, X^{\mathbf{u}}(s)))ds \\ & \qquad +\sigma(s, X^{\mathbf{u}}(s), \mathbf{u}(s, X^{\mathbf{u}}(s)))dW(s), \ s \in [t, T], \\ & X^{\mathbf{u}}(t) = x, \end{cases}$$

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  - $W(t) := \big(W_1(t), ..., W_d(t)\big)^\top$, $t \geq 0$ is a standard d-dimensional Brownian motion
  - The controlled diffusion process $X^{\mathbf{u}}$ under $\mathbf{u}$ takes values in $\mathbb{X}$, which is either $\mathbb{R}$ or $(0, +\infty)$.

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- The agent's objective function at time t is given as follows:

$$
\begin{aligned}
J(t, x; \mathbf{u}) = \mathbb{E}_{t,x}\Big[\int_t^T C\big(t, x, s, X^{\mathbf{u}}(s), \mathbf{u}(s, X^{\mathbf{u}}(s))\big) ds \\
+ F\big(t, x, X^{\mathbf{u}}(T)\big)\Big] + G\big(t, x, \mathbb{E}_{t,x}[X^{\mathbf{u}}(T)]\big).
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- Time-inconsistency arises from the dependence of C, F, and G on (t,x) and from the nonlinear dependence on $\mathbb{E}_{t,x}[X^{\mathbf{u}}(T)]$.

Standing Assumption

(i) μ, σ, C, F, and G are measurable. For any fixed $u \in \mathbb{U}$ and $x \in \mathbb{X}$, $\mu(t,x,u)$ and $\sigma(t,x,u)$ are right-continuous in $t \in [0,T)$. For any fixed $u \in \mathbb{U}$, $(\tau,y) \in [0,T) \times \mathbb{X}$, and $(t,x) \in [0,T) \times \mathbb{X}$, $\lim_{t'>t,(t',x')\to(t,x)} C(\tau,y,t',x',u) = C(\tau,y,t,x,u)$.

(ii) For any fixed $u \in \mathbb{U}$, $\mu(t,x,u)$ and $\sigma(t,x,u)$ are locally Lipschitz in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$.

(iii) For any fixed $u \in \mathbb{U}$, when $\mathbb{X} = \mathbb{R}$, $\mu(t,x,u)$ and $\sigma(t,x,u)$ are of linear growth in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$, and when $\mathbb{X} = (0,+\infty)$, they have bounded norm in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$.

(iv) For any fixed $u \in \mathbb{U}$ and $(\tau,y) \in [0,T) \times \mathbb{X}$, $C(\tau,y,t,x,u)$ and $F(\tau,y,x)$ are of polynomial growth in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$.

(v) For each fixed $(\tau,y) \in [0,T) \times \mathbb{X}$, $G(\tau,y,z)$ is twice continuously differentiable with respect to z, and the first- and second-order derivatives are denoted as G_z and G_{zz}, respectively.

Feasible Strategies

Definition

A feedback strategy $\mathbf{u}$ is *feasible* if the following hold:

(i) For any fixed $x \in \mathbb{X}$, $\mu^{\mathbf{u}}$ and $\sigma^{\mathbf{u}}$ are right-continuous in $t \in [0,T)$. For any fixed $(\tau, y) \in [0,T) \times \mathbb{X}$ and $(t,x) \in [0,T) \times \mathbb{X}$, $\lim_{t'>t,(t',x')\to(t,x)} C^{\tau,y,\mathbf{u}}(t',x') = C^{\tau,y,\mathbf{u}}(t,x)$.

(ii) $\mu^{\mathbf{u}}$, $\sigma^{\mathbf{u}}$ are locally Lipschitz in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$.

(iii) When $\mathbb{X} = \mathbb{R}$, $\mu^{\mathbf{u}}$ and $\sigma^{\mathbf{u}}$ are of linear growth in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$, and when $\mathbb{X} = (0,+\infty)$, they have bounded norm in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$.

(iv) For each fixed $(\tau, y) \in [0,T) \times \mathbb{X}$,
$\mathbb{E}_{t,x}\left[\sup_{s\in[t,T]} |C^{\tau,y,\mathbf{u}}(s, X^{\mathbf{u}}(s))| + |F(\tau, y, X^{\mathbf{u}}(T))| + |X^{\mathbf{u}}(T)|\right]$ is of polynomial growth in $x \in \mathbb{X}$, uniformly in $t \in [0,T]$.

Denote the set of feasible strategies as $\mathbf{U}$.

Feasibility of Constant Strategies

Lemma

Let the Standing Assumption hold. Then, all constant strategies are feasible, i.e.,
$\mathbf{U} \supseteq \mathbf{U}_0 := \{\mathbf{u} \mid \exists u \in \mathbb{U}$ *such that* $\mathbf{u}(t,x) = u,\ \forall t \in [0,T], x \in \mathbb{X}\}$.

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- For given $t \in [0, T)$, $\epsilon \in (0, T-t)$, $\hat{\mathbf{u}} \in \mathbf{U}$, and $\mathbf{a} \in \mathbf{D}$, define

$$\mathbf{u}_{t,\epsilon,\mathbf{a}}(s, y) := \begin{cases} \mathbf{a}(s, y), & s \in [t, t+\epsilon), y \in \mathbb{X} \\ \hat{\mathbf{u}}(s, y), & s \notin [t, t+\epsilon), y \in \mathbb{X}. \end{cases}$$

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- $\mathbf{u}_{t,\epsilon,\mathbf{a}}$ is also feasible, i.e., $\mathbf{u}_{t,\epsilon,\mathbf{a}} \in \mathbf{U}$.

Weak Equilibrium Strategy

- For a given initial state x_0 and a given strategy $\hat{\mathbf{u}}$, denote by $\mathbb{X}_t$ the set of **reachable states at time** t, i.e., $\mathbb{X}_t$ is the union of the interior of the support and the atoms of the distribution of $X^{\hat{\mathbf{u}}}(t)$ conditional on the information at time 0.

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Definition

Let $\hat{\mathbf{u}} \in \mathbf{U}$ and denote by $\mathbb{X}_t$ the set of reachable states at time t by $\hat{\mathbf{u}}$. $\hat{\mathbf{u}}$ is a **weak equilibrium strategy** if for any $t \in [0, T)$, $x \in \mathbb{X}_t$, and any $\mathbf{a} \in \mathbf{D}$, we have

$$\liminf_{\epsilon \downarrow 0} \frac{J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) - J(t, x; \hat{\mathbf{u}})}{\epsilon} \leq 0.$$

Reachable States

- The above notion of weak equilibrium strategies is based on the first-order condition and thus essentially the same as the ones used in the literature, except the following difference:

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    - In the ones used in the literature, the first-order condition needs to hold for any t and $x \in \mathbb{X}$;
- Although the agent's self today cannot control the action of her future selves, her action today actually determines the state process in the future based on which her future selves make decisions.
- Thus, when examining whether a chosen strategy is equilibrium, the actions of the future selves in the states that are not reachable are irrelevant.

Regular and Strong Equilibrium Strategies

Definition

Let $\hat{\mathbf{u}} \in \mathbf{U}$ and denote by $\mathbb{X}_t$ the set of reachable states at time t by $\hat{\mathbf{u}}$. $\hat{\mathbf{u}}$ is a **strong equilibrium strategy** if for any $t \in [0, T)$, $x \in \mathbb{X}_t$, and any $\mathbf{a} \in \mathbf{D}$, there exists $\epsilon_0 \in (0, T - t)$ such that

$$J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) - J(t, x; \hat{\mathbf{u}}) \leq 0, \quad \forall \epsilon \in (0, \epsilon_0].$$

Definition

Let $\hat{\mathbf{u}} \in \mathbf{U}$ and denote by $\mathbb{X}_t$ the set of reachable states at time t by $\hat{\mathbf{u}}$. $\hat{\mathbf{u}}$ is a **regular equilibrium strategy** if for any $t \in [0, T)$, $x \in \mathbb{X}_t$, and any $\mathbf{a} \in \mathbf{D}$ with $\mathbf{a}(t, x) \neq \hat{\mathbf{u}}(t, x)$, there exists $\epsilon_0 \in (0, T - t)$ such that

$$J(t, x; \mathbf{u}_{t,\epsilon,\mathbf{a}}) - J(t, x; \hat{\mathbf{u}}) \leq 0, \quad \forall \epsilon \in (0, \epsilon_0].$$

Two Observations

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Two Observations

- If $\hat{\mathbf{u}}$ is a strong equilibrium strategy, it is also a regular equilibrium strategy and a weak equilibrium strategy.
- In all the above three notions of equilibria,
  - the larger the set $\mathbf{D}$ is, the stronger requirement for an equilibrium strategy is; and
  - when $\mathbb{X}_t$ is replaced by a larger set, the requirement for an equilibrium strategy is stronger.

Outline

Notations

- Here and hereafter, $\hat{\mathbf{u}} \in \mathbf{U}$ denotes a given strategy.
- Recall the agent's objective function at time t

$$J(t,x;\mathbf{u}) = \mathbb{E}_{t,x}\Big[\int_t^T C\big(t,x,s,X^{\mathbf{u}}(s),\mathbf{u}(s,X^{\mathbf{u}}(s))\big)ds \\ + F\big(t,x,X^{\mathbf{u}}(T)\big)\Big] + G\big(t,x,\mathbb{E}_{t,x}[X^{\mathbf{u}}(T)]\big),$$

  and denote $C^{\tau,y,\mathbf{u}}(t,x) := C\big(\tau,y,t,x,\mathbf{u}(t,x)\big)$.
- For each fixed $(\tau,y) \in [0,T) \times \mathbb{X}$, denote

$$f^{\tau,y}(t,x) := \mathbb{E}_{t,x}[F(\tau,y,X^{\hat{\mathbf{u}}}(T))],\ g(t,x) := \mathbb{E}_{t,x}[X^{\hat{\mathbf{u}}}(T)],\ t \in [0,T], x \in \mathbb{X}.$$

- For fixed $(\tau,y) \in [0,T) \times \mathbb{X}$ and $s \in [0,T]$, denote

$$c^{\tau,y,s}(t,x) := \mathbb{E}_{t,x}[C^{\tau,y,\hat{\mathbf{u}}}(s,X^{\hat{\mathbf{u}}}(s))],\ t \in [0,s], x \in \mathbb{X}.$$

1st-Order Smoothness Assumption

Assumption

For any fixed $(\tau, y) \in [0, T) \times \mathbb{X}$,

(i) $f^{\tau,y}$ *and* g *are in* $\mathfrak{C}^{1,2}([0,T] \times \mathbb{X})$ *and for each fixed* $s \in [0,T]$, $c^{\tau,y,s}$ *is in* $\mathfrak{C}^{1,2}([0,s] \times \mathbb{X})$;

(ii) $f^{\tau,y}(t,x)$, $f_t^{\tau,y}(t,x)$, $f_x^{\tau,y}(t,x)$, $f_{xx}^{\tau,y}(t,x)$, $g(t,x)$, $g_t(t,x)$, $g_x(t,x)$, *and* $g_{xx}(t,x)$ *are of polynomial growth in* $x \in \mathbb{X}$, *uniformly in* $t \in [0,T]$;

(iii) $c^{\tau,y,s}(t,x)$, $c_t^{\tau,y,s}(t,x)$, $c_x^{\tau,y,s}(t,x)$, *and* $c_{xx}^{\tau,y,s}(t,x)$ *are of polynomial growth in* $x \in \mathbb{X}$, *uniformly in* $t \in [0,s]$ *and* $s \in [0,T]$.

First-Order Expansion

Lemma

Suppose the Standing and the 1st-Order Smoothness Assumptions hold. Then, for any fixed $(t,x) \in [0,T) \times \mathbb{X}$ *and* $\mathbf{a} \in \mathbf{U}$*, we have*

$$J(t,x;\mathbf{u}_{t,\epsilon,\mathbf{a}}) - J(t,x;\hat{\mathbf{u}}) = \epsilon \Gamma^{\hat{\mathbf{u}}}(t,x;\mathbf{a}) + \epsilon o(1),$$

where

$$\Gamma^{\hat{\mathbf{u}}}(t,x;\mathbf{a}) := C^{t,x,\mathbf{a}}(t,x) - C^{t,x,\hat{\mathbf{u}}}(t,x) + \int_t^T \mathcal{A}^{\mathbf{a}} c^{t,x,s}(t,x)ds$$
$$+\mathcal{A}^{\mathbf{a}} f^{t,x}(t,x) + G_z(t,x,g(t,x))\mathcal{A}^{\mathbf{a}} g(t,x).$$

Moreover, $\Gamma^{\hat{\mathbf{u}}}(t,x;\mathbf{a}) = \Gamma^{\hat{\mathbf{u}}}(t,x;\tilde{\mathbf{a}})$ *for any* $\mathbf{a}, \tilde{\mathbf{a}} \in \mathbf{U}$ *with* $\mathbf{a}(t,x) = \tilde{\mathbf{a}}(t,x)$ *and in particular,* $\Gamma^{\hat{\mathbf{u}}}(t,x;\mathbf{a}) = 0$ *if* $\mathbf{a}(t,x) = \hat{\mathbf{u}}(t,x)$.

First-Order Analysis of Equilibrium Strategies

Theorem

Suppose the Standing and the 1st-Order Smoothness Assumptions hold and $\mathbf{U}_0 \subseteq \mathbf{D} \subseteq \mathbf{U}$. *Then, the following are true:*

(i) $\hat{\mathbf{u}}$ *is a weak equilibrium strategy if and only if*

$$\Gamma^{\hat{\mathbf{u}}}(t,x;u) \le 0,\ \forall u \in \mathbb{U}, x \in \mathbb{X}_t, t \in [0,T).$$

(ii) *If* $\hat{\mathbf{u}}$ *is a regular equilibrium strategy, it is also a weak equilibrium strategy.*

(iii) *If*

$$\Gamma^{\hat{\mathbf{u}}}(t,x;u) < 0,\ \forall u \in \mathbb{U} \text{ with } u \neq \hat{\mathbf{u}}(t,x), x \in \mathbb{X}_t, t \in [0,T),$$

then $\hat{\mathbf{u}}$ *is a regular equilibrium strategy.*

(iv) *Suppose that for any* $x \in \mathbb{X}_t$ *and* $t \in [0,T)$, *the maximization of* $\Gamma^{\hat{\mathbf{u}}}(t,x;u)$ *in* u *admits a unique maximizer, which in particular holds when* $\mathbb{U}$ *is a convex set and* $\Gamma^{\hat{\mathbf{u}}}(t,x;u)$ *is strictly concave in* u. *Then,* $\hat{\mathbf{u}}$ *is a weak equilibrium strategy if and only if it is a regular equilibrium strategy.*

Further Notations

- In the following, suppose $\mathbb{U} \subseteq \mathbb{R}^m$.
- Denote

$$\mathbf{U}_S := \{\mathbf{u} \in \mathbf{U} \cap \mathfrak{C}_m^{1,2}([0,T] \times \mathbb{X}) \mid \mathbf{u}, \mathbf{u}_t, \mathbf{u}_x, \mathbf{u}_{xx} \text{ are of polynomial growth in } x \in \mathbb{X}, \text{ uniformly in } t \in [0,T]\}.$$

- With the Standing Assumption, we have $\mathbf{U}_0 \subseteq \mathbf{U}_S$.
- In the following, we consider a chosen strategy $\hat{\mathbf{u}} \in \mathbf{U}_S$.

2nd-Order Smoothness Assumption

Assumption

$\mathbb{U} \subseteq \mathbb{R}^m$, $\hat{\mathbf{u}} \in \mathbf{U}_S$, *the 1st-Order Assumption holds, and the following hold:*

(i) *For each* $\xi \in \{\mu, \|\sigma\|^2\}$, $\xi \in \mathfrak{C}^{1,2,2}([0,T] \times \mathbb{X} \times \mathbb{U})$, *and* ξ, ξ_t, ξ_x, ξ_u, ξ_{xx}, ξ_{uu}, *and* ξ_{xu} *are of polynomial growth in* $(x,u) \in \mathbb{X} \times \mathbb{U}$, *uniformly in* $t \in [0,T]$.

(ii) *Given any* $(\tau, y) \in [0,T) \times \mathbb{R}$, *for any* $\xi \in \{f_t^{\tau,y}, f_x^{\tau,y}, f_{xx}^{\tau,y}, g_t, g_x, g_{xx}\}$, $\xi \in \mathfrak{C}^{1,2}([0,T] \times \mathbb{X})$ *and* ξ_t, ξ_x, *and* ξ_{xx} *are of polynomial growth in* $x \in \mathbb{X}$, *uniformly in* $t \in [0,T]$.

(iii) *Given any* $(\tau, y) \in [0,T) \times \mathbb{R}$, *for any* $\xi^s \in \{c_t^{\tau,y,s}, c_x^{\tau,y,s}, c_{xx}^{\tau,y,s}\}$, $\xi^s \in \mathfrak{C}^{1,2}([0,s] \times \mathbb{X})$ *for each fixed* $s \in [0,T]$ *and* ξ_t^s, ξ_x^s, *and* ξ_{xx}^s *are of polynomial growth in* $x \in \mathbb{X}$, *uniformly in* $t \in [0,s]$ *and* $s \in [0,T]$.

(iv) *Given any* $(\tau, y) \in [0,T) \times \mathbb{R}$ *and* $(t,x) \in [0,T) \times \mathbb{R}$, *for* $\xi^{\tau,y,s}(t,x)$ *to be any of* $c_t^{\tau,y,s}(t,x)$, $c_x^{\tau,y,s}(t,x)$, *and* $c_{xx}^{\tau,y,s}(t,x)$, $\lim_{s>t, s\downarrow t} \xi^{\tau,y,s}(t,x) = \xi^{\tau,y,t}(t,x)$.

Second-Order Expansion

Lemma

Suppose the Standing and the 2nd-Order Smoothness Assumption hold. Then, for any $(t,x)\in[0,T)\times\mathbb{X}$ *and* $\mathbf{a}\in\mathbf{U}_S$*, we have*

$$J(t,x;\mathbf{u}_{t,\epsilon,\mathbf{a}})-J(t,x;\hat{\mathbf{u}})=\epsilon\Gamma^{\hat{\mathbf{u}}}(t,x;\mathbf{a})+\frac{1}{2}\epsilon^2\Lambda^{\hat{\mathbf{u}}}(t,x;\mathbf{a})+\epsilon^2 o(1),$$

where $\Gamma^{\hat{\mathbf{u}}}(t,x;\mathbf{a})$ *is given as in the first-order expansion and*

$$\begin{aligned}\Lambda^{\hat{\mathbf{u}}}(t,x;\mathbf{a}):=&\,\mathcal{A}^{\mathbf{a}}C^{t,x,\mathbf{a}}(t,x)-\mathcal{A}^{\hat{\mathbf{u}}}C^{t,x,\hat{\mathbf{u}}}(t,x)-2\mathcal{A}^{\mathbf{a}}c^{t,x,t}(t,x)\\&+\int_t^T(\mathcal{A}^{\mathbf{a}})^2c^{t,x,s}(t,x)ds+(\mathcal{A}^{\mathbf{a}})^2f^{t,x}(t,x)\\&+G_z(t,x,g(t,x))(\mathcal{A}^{\mathbf{a}})^2g(t,x)+G_{zz}(t,x,g(t,x))\left(\mathcal{A}^{\mathbf{a}}g(t,x)\right)^2.\end{aligned}$$

Second-Order Analysis of Equilibrium Strategies

Theorem

Consider a weak equilibrium strategy $\hat{\mathbf{u}}$*, and suppose the Standing and the 2nd-Order Smoothness Assumption hold and* $\mathbf{D} \subseteq \mathbf{U}_S$*. Define*

$$\mathcal{R}^{\hat{\mathbf{u}}} := \{(t,x,u) \in [0,T) \times \mathbb{X} \times \mathbb{U} \mid x \in \mathbb{X}_t, \Gamma^{\hat{\mathbf{u}}}(t,x;u) = 0\},$$
$$\mathcal{R}_0^{\hat{\mathbf{u}}} := \{(t,x,u) \in [0,T) \times \mathbb{X} \times \mathbb{U} \mid x \in \mathbb{X}_t, \Gamma^{\hat{\mathbf{u}}}(t,x;u) = 0, u \neq \hat{\mathbf{u}}(t,x)\}.$$

The following are true:

(i) *If* $\mathcal{R}_0^{\hat{\mathbf{u}}} = \emptyset$ *or if* $\Lambda^{\hat{\mathbf{u}}}(t,x;\mathbf{a}) < 0$ *for all* $x \in \mathbb{X}_t$*,* $t \in [0,T)$*, and* $\mathbf{a} \in \mathbf{D}$ *with* $(t,x,\mathbf{a}(t,x)) \in \mathcal{R}_0^{\hat{\mathbf{u}}}$*, then* $\hat{\mathbf{u}}$ *is a regular equilibrium strategy. If* $\Lambda^{\hat{\mathbf{u}}}(t,x;\mathbf{a}) > 0$ *for some* $t \in [0,T)$*,* $x \in \mathbb{X}_t$*, and* $\mathbf{a} \in \mathbf{D}$ *with* $(t,x,\mathbf{a}(t,x)) \in \mathcal{R}_0^{\hat{\mathbf{u}}}$*, then* $\hat{\mathbf{u}}$ *is not a regular equilibrium strategy.*

(ii) *If* $\Lambda^{\hat{\mathbf{u}}}(t,x;\mathbf{a}) < 0$ *for all* $x \in \mathbb{X}_t$*,* $t \in [0,T)$*, and* $\mathbf{a} \in \mathbf{D}$ *with* $(t,x,\mathbf{a}(t,x)) \in \mathcal{R}^{\hat{\mathbf{u}}}$*, then* $\hat{\mathbf{u}}$ *is a strong equilibrium strategy. If* $\Lambda^{\hat{\mathbf{u}}}(t,x;\mathbf{a}) > 0$ *for some* $t \in [0,T)$*,* $x \in \mathbb{X}_t$*, and* $\mathbf{a} \in \mathbf{D}$ *with* $(t,x,\mathbf{a}(t,x)) \in \mathcal{R}^{\hat{\mathbf{u}}}$*, then* $\hat{\mathbf{u}}$ *is not a strong equilibrium strategy.*

Outline

Mean-Variance Problem in Björk et al. (2014)

$$\begin{cases} \max\limits_{\mathbf{u}} & \mathbb{E}_{t,x}[X^{\mathbf{u}}(T)] - \frac{\gamma}{2x}\mathrm{var}_{t,x}(X^{\mathbf{u}}(T)) \\ \text{subject to} & dX^{\mathbf{u}}(s) = \big(rX^{\mathbf{u}}(s) + b\mathbf{u}(s, X^{\mathbf{u}}(s))\big)ds \\ & \quad +\bar{\sigma}\mathbf{u}(s, X^{\mathbf{u}}(s))dW(s),\ s \in [t, T],\ X^{\mathbf{u}}(t) = x, \end{cases}$$

where

- constant r: risk-free rate
- constant $b \neq 0$: mean excess return rate of a stock
- constant $\bar{\sigma} > 0$: stock's volatility
- $x \in (0, +\infty)$: agent's wealth level at time t
- $\mathbf{u}(s, X^{\mathbf{u}}(s))$ is the dollar amount invested into a stock at time s
- $X^{\mathbf{u}}(s)$ is the agent's wealth at time s
- $\gamma > 0$ is the agent's risk aversion degree

Mean-Variance Problem in Björk et al. (2014) (Cont'd)

Proposition

Suppose that the agent has wealth $x_0 > 0$ *at time 0. Define*

$$\hat{\mathbf{u}}(t,x) = x\theta(t),\ t \in [0,T], x \in \mathbb{R},$$

where $\theta(t)$ *solves following equation*

$$\theta(t) = \frac{b}{\gamma\bar{\sigma}^2}\left(e^{-\int_t^T\left(r+b\theta(s)+\bar{\sigma}^2\theta(s)^2\right)ds} + \gamma e^{-\int_t^T \bar{\sigma}^2\theta(s)^2 ds} - \gamma\right),\ t \in [0,T].$$

Then, the Standing and 2nd-Order Smoothness Assumptions hold, and the following are true:

(i) $\hat{\mathbf{u}}$ *is both a weak and a regular equilibrium strategy for any* $\mathbf{D} \subseteq \mathbf{U}$.

(ii) $\hat{\mathbf{u}}$ *is not a strong equilibrium strategy for any* $\mathbf{D} \supseteq \mathbf{U}_0$.

Mean-Variance Problem in Basak and Chabakauri (2010)

$$\begin{cases} \max\limits_{\mathbf{u}} & J(t,x;\mathbf{u}) := \mathbb{E}_{t,x}[X^{\mathbf{u}}(T)] - \frac{\gamma}{2}\mathrm{var}_{t,x}(X^{\mathbf{u}}(T)) \\ \text{subject to} & dX^{\mathbf{u}}(s) = \big(rX^{\mathbf{u}}(s) + b\mathbf{u}(s,X^{\mathbf{u}}(s))\big)ds \\ & \quad +\bar{\sigma}\mathbf{u}(s,X^{\mathbf{u}}(s))dW(s),\ s\in[t,T],\ X^{\mathbf{u}}(t)=x, \end{cases}$$

where

- constant r: risk-free rate
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- constant $\bar{\sigma} > 0$: stock's volatility
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- $\mathbf{u}(s, X^{\mathbf{u}}(s))$ is the dollar amount invested into a stock at time s
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- $\gamma > 0$ is the agent's risk aversion degree

Mean-Variance Problem in Basak and Chabakauri (2010) (Cont'd)

Proposition

Define

$$\hat{\mathbf{u}}(t,x) = \frac{1}{\gamma}\frac{b}{\bar{\sigma}^2}e^{-r(T-t)},\ t \in [0,T], x \in \mathbb{R}.$$

Then, the Standing and 2nd-Order Smoothness Assumptions hold, and the following are true:

(i) $\hat{\mathbf{u}}$ *is both a weak and a regular equilibrium strategy for any* $\mathbf{D} \subseteq \mathbf{U}$.

(ii) $\hat{\mathbf{u}}$ *is a strong equilibrium strategy for* $\mathbf{D} = \mathbf{U}_0$.

(iii) $\hat{\mathbf{u}}$ *is not a strong equilibrium strategy for any* $\mathbf{D} \supseteq \mathbf{U}_S$.

Investment and consumption in Ekeland and Pirvu (2008)

$$\begin{cases} \max_{\mathbf{u}} & \mathbb{E}_{t,x}\left[\int_t^T h(s-t)\frac{(\zeta(s,X^{\mathbf{u}}(s))X^{\mathbf{u}}(s))^{1-\gamma}}{1-\gamma}ds + h(T-t)\frac{(X^{\mathbf{u}}(T))^{1-\gamma}}{1-\gamma}\right] \\ \text{subject to} & dX^{\mathbf{u}}(s) = X^{\mathbf{u}}(s)\Big[\big(r + b\theta(s,X^{\mathbf{u}}(s)) - \zeta(s,X^{\mathbf{u}}(s))\big)ds \\ & \quad +\bar{\sigma}\theta(s,X^{\mathbf{u}}(s))dW(s)\Big],\ s\in[t,T],\ X^{\mathbf{u}}(t) = x, \end{cases}$$

where

- constant r: risk-free rate
- constant $b \neq 0$: mean return rate of a stock
- constant $\bar{\sigma} > 0$: stock's volatility
- $x \in (0, +\infty)$: agent's wealth level at time t
- $\mathbf{u}(s, X^{\mathbf{u}}(s)) = (\zeta(s, X^{\mathbf{u}}(s)), \theta(s, X^{\mathbf{u}}(s)))$ with ζ standing for consumption and θ standing for investment.
- $X^{\mathbf{u}}(s)$ is the agent's wealth at time s
- $\gamma > 0$ is the agent's relative risk aversion degree
- $h(\tau)$ is the discounting function

Investment and consumption in Ekeland and Pirvu (2008) (Cont'd)

Proposition

Define

$$\hat{\mathbf{u}}(t,x) = \left(\hat{\zeta}(t,x), \hat{\theta}(t,x)\right) := \left(k(t)^{-\frac{1}{\gamma}}, \frac{b}{\gamma\bar{\sigma}^2}\right),\ t\in[0,T), x\in(0,+\infty),$$

$$k(t) = \int_t^T h(s-t)e^{(1-\gamma)\left(r+b^2/(2\gamma\bar{\sigma}^2)\right)(s-t)}k(s)^{-\frac{1-\gamma}{\gamma}}e^{-(1-\gamma)\int_t^s k(z)^{-1/\gamma}dz}ds$$
$$+ h(T-t)e^{(1-\gamma)\left(r+b^2/(2\gamma\bar{\sigma}^2)\right)(T-t)}e^{-(1-\gamma)\int_t^T k(s)^{-1/\gamma}ds},\ t\in[0,T].$$

Then, the Standing and 2nd-Order Smoothness Assumptions hold, and the following are true:

(i) $\hat{\mathbf{u}}$ *is both a weak and a regular equilibrium strategy for any* $\mathbf{D}\subseteq\mathbf{U}$.

(ii) $\hat{\mathbf{u}}$ *is not a strong equilibrium strategy for any* $\mathbf{D}\supseteq\mathbf{U}_0$.

Optimal Consumption with A Bequest

$$\begin{cases} \max\limits_{\mathbf{u}} & \mathbb{E}_{t,x}\left[\int_t^T h(s-t)\mathbf{u}(s,X^{\mathbf{u}}(s))ds + \tilde{h}(T-t)X^{\mathbf{u}}(T)\right] \\ \text{subject to} & dX^{\mathbf{u}}(s) = \big(b-\mathbf{u}(s,X^{\mathbf{u}}(s))\big)ds + \bar{\sigma}dW(s),\ s\in[t,T], \\ & X^{\mathbf{u}}(t) = x, \end{cases}$$

where

- $bdt+\bar{\sigma}dW(t)$: agent's endowment at each instant $t\in[0,T]$
- $\mathbf{u}(s,X^{\mathbf{u}}(s))ds$: agent's consumption at time s
- $X^{\mathbf{u}}(s)$: agent's wealth at time s
- $h(\tau)$: discount function for consumption
- $\tilde{h}(\tau)$: discount function for the bequest

Optimal Consumption with A Bequest (Cont'd)

Assumption

Suppose that h *and* $\tilde{h}$ *are continuously differentiable on* $[0,T]$ *with* $h(0)=\tilde{h}(0)=1$, *that* h' *is absolutely continuous on* $[0,T]$ *with a bounded density* h'', *that* $h'(0)\neq 0$ *and* $\tilde{h}'(0)=0$, *and that* ψ, *which is the unique solution to the following equation:*

$$\psi(t)=\frac{1}{h'(0)}\left[h'(T-t)-\tilde{h}'(T-t)-\int_t^T\psi(s)h''(s-t)ds\right],\ t\in[0,T],$$

satisfies $\psi(t)>0$ *for all* $t\in[0,T]$.

The above assumption is satisfied for some commonly used discount functions with reasonable parameter values

Optimal Consumption with A Bequest (Cont'd)

Proposition

Suppose the above assumption holds. Fix any constant $b_0 \in \mathbb{R}$ *and denote*

$$\hat{\mathbf{u}}(t,x) := b_0 + k(t)x,\ t \in [0,T], x \in \mathbb{R},$$

where $k(t) = -\psi'(t)/\psi(t), t \in [0,T]$ *with* $\psi(t)$ *as in the above assumption. Then, the Standing Assumption and the 2nd-Order Smoothness Assumption hold, and the following are true:*

(i) $\hat{\mathbf{u}}$ *is a weak equilibrium strategy for any* $\mathbf{D} \subseteq \mathbf{U}$.

(ii) *For any* $\mathbf{D} \supseteq \mathbf{U}_0$, $\hat{\mathbf{u}}$ *is neither a regular nor a strong equilibrium strategy.*

Outline

Conclusions

- The notion of weak equilibrium strategies employed in the literature to study continuous-time time-inconsistent decision problems is not aligned with the standard definition of equilibria in the game theory.
- To address the above issue, we proposed two new notions of equilibrium strategies: regular and strong equilibria.
- We derived sufficient conditions as well as necessary conditions for a strategy to be a regular equilibrium and to be a strong equilibrium.
- We examined three time-inconsistent portfolio selection problems in the literature, and show that the weak equilibrium strategies derived therein are also regular equilibria but are not strong equilibria.
- We further provide an example to show that a weak equilibrium strategy may not be a regular equilibrium.

Conclusions (Cont'd)

- Summary of the findings:

$$\text{strong equilibrium} \Rightarrow \text{regular equilibrium} \Rightarrow \text{weak equilibrium}$$

$$\text{strong equilibrium} \not\Leftarrow \text{regular equilibrium} \not\Leftarrow \text{weak equilibrium}$$

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