Fractional Brownian motion with zero Hurst parameter: a rough volatility viewpoint

Eyal Neuman

Imperial College London

Joint work with Mathieu Rosenbaum

May 21, 2019

Fractional Brownian motion

1. A fBm $(B_t^H)_{t\in\mathbb{R}}$ with Hurst parameter $H \in (0,1)$ is a zero-mean Gaussian process with covariance kernel given by

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

2. It has stationary increments and is self-similar with parameter H, that is $(B_{at}^H)_{t\in\mathbb{R}}$ has the same law as $(a^H B_t^H)_{t\in\mathbb{R}}$ for any $a > 0$.

Fractional Brownian motion

3. Sample paths of fBm have almost surely Hölder regularity $H - \varepsilon$ for any $\varepsilon > 0$.

4. Long memory property of the increments when $H > 1/2$. This means that for $H > 1/2$, we have

$$\sum_{i=1}^{+\infty} \mathrm{Cov}[(B_{i+1}^H - B_i^H), B_1^H] = +\infty,$$

which is useful for modeling persistent phenomena.

Applications of Fractional Brownian motion

Fractional Brownian motion is a very popular modeling object in many fields:

- Hydrology, see for example Molz, Liu, Szulga (1997),
- Telecommunications and network traffic: Leland et al. (1994), Mikosch et al. (2002).
- Finance, seminal paper by Comte and Renault (1998).

Rough-Volatility Models in Finance

1. Recently, Gatheral et al. (2014) performed a careful analysis of financial time series.
2. They suggested that the log-volatility process behaves like a fBm with $H \approx 0.1$ **(even more recently Fukasawa et al. estimated $H \approx 0.06$)**.
3. Various approaches using a fBm with small Hurst parameter have been introduced for volatility modeling.
4. These models are referred to as *rough volatility models*

Rough-Volatility Models in Finance

Some of the people involved: Alos, Bayer, Bennedsen, ,El Euch, Forde, Friz, Fukasawa, Gassiat, Gatheral, Gulisashvili, Harms, Horvath, Jacquier, Martini, Pakkanen, Pallavicini, Podolskij, Rosenbaum, Stemper, Zhang...

5. Such small estimated values for H (between 0.05 and 0.2) have been found when studying the volatility process of thousands of assets (Bennedsen et al. '17).

6. A natural question is the behavior of the fBm in the limiting case when $H \to 0$.

7. Of course, putting directly $H = 0$ in the covariance

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

does not lead to a relevant process.

Conjecture

FBm B^H, after suitable renormalization, converges to a log-correlated Gaussian field as $H \to 0$. That is, to a centred Gaussian field with the "covariance kernel"

$$C(s,t) \sim \log_+ \frac{1}{|t-s|}.$$

Here $\log_+(u) = \max(\log(u), 0)$.

Log-Correlated Gaussian Fields

1. Let $\mathcal{S}$ the real Schwartz space.
2. We write $\mathcal{S}'$ for the dual of $\mathcal{S}$, that is the space of tempered distributions.
3. We also define the subspace $\mathcal{S}_0$ of the real Schwartz space, consisting of functions ϕ from $\mathcal{S}$ with $\int_{\mathbb{R}} \phi(s)\, ds = 0$, and its topological dual $\mathcal{S}'/\mathbb{R}$.

Log-Correlated Gaussian Fields

4. A log-correlated Gaussian field (LGF for short) $X \in \mathcal{S}'/\mathbb{R}$, is a centered Gaussian field whose covariance kernel satisfies

$$\mathbb{E}[\langle X, \phi_1 \rangle \langle X, \phi_2 \rangle] = \int_{\mathbb{R}} \int_{\mathbb{R}} \log \frac{1}{|t-s|} \phi_1(t) \phi_2(s) \, dt \, ds,$$

for any $\phi_1, \phi_2 \in \mathcal{S}_0$.

Log-Correlated Gaussian Fields

1. LGFs are closely related to some multifractal processes (Mandelbrot et al. '97, Barral '02, Bacry and Muzy '03).

2. A process $(Y_t)_{t\geq 0}$ is said to be multifractal if for a range of values of q, we have for some $T > 0$

$$\mathbb{E}\big[|Y_{t+\ell} - Y_t|^q\big] \sim C(q)\ell^{\zeta(q)}, \quad \text{for } 0 < \ell \leq T,$$

3. where $C(q) > 0$ is a constant and $\zeta(\cdot)$ is a non-linear concave function.

Multifractal Random Walk

1. Multifractal random walk model for the log-price of an asset (Bacry et al. '01) is defined as

$$Y_t = B_{M([0,t])},$$

2. where B is a Brownian motion and

$$M(t) = \lim_{l \to 0} \sigma^2 \int_0^t e^{w_l(u)} du, \text{ a.s.,}$$

with $\sigma^2 > 0$.

1. w_l a Gaussian process such that for some $\lambda^2 > 0$ and $T > 0$

$$\text{Cov}[w_l(t), w_l(t')] = \lambda^2 \text{log}(T/|t - t'|), \text{ for } l < |t - t'| \leq T,$$

2. Hence we see that M formally corresponds to a measure of the form $\exp(X_t)dt$, where X is a LGF.

1. w_l a Gaussian process such that for some $\lambda^2 > 0$ and $T > 0$

$$\mathrm{Cov}[w_l(t), w_l(t')] = \lambda^2 \log(T/|t-t'|), \text{ for } l < |t-t'| \leq T,$$

2. Hence we see that M formally corresponds to a measure of the form $\exp(X_t)dt$, where X is a LGF.

3. The precise definition of such measures: Kahane '85 and the generalizations of Gaussian multiplicative chaos by Rhodes and Vargas '14, '16.

4. LGFs and Gaussian multiplicative chaos have an extensive use in turbulence, disordered systems and Liouville quantum gravity.

Convergence of fBm towards a LGF

1. In our main theorem we prove an accurate statement about the convergence of normalized fBm towards a LGF as H goes to zero.

2. Our normalized sequence of processes $(X_{\cdot}^{H})_{H\in(0,1)}$ is defined through

$$X_t^H = \frac{B_t^H - \frac{1}{t}\int_0^t B_u^H \, du}{\sqrt{H}}, \quad t \in \mathbb{R}.$$

Convergence of fBm towards a LGF

1. In our main theorem we prove an accurate statement about the convergence of normalized fBm towards a LGF as H goes to zero.

2. Our normalized sequence of processes $(X_\cdot^H)_{H\in(0,1)}$ is defined through

$$X_t^H = \frac{B_t^H - \frac{1}{t}\int_0^t B_u^H \, du}{\sqrt{H}}, \quad t \in \mathbb{R}.$$

3. We say that X^H converges weakly to X, as H tends to 0, if for any $\phi \in \mathcal{S}$ we have

$$\langle X^H, \phi\rangle \to \langle X, \phi\rangle,$$

in law, as H tends to 0.

Theorem (N. and Rosenbaum, 2018)

The sequence $\{X_t^H\}_{t\in\mathbb{R}}$ converges weakly as H tends to zero towards a centered Gaussian field X satisfying for any $\phi_1, \phi_2 \in \mathcal{S}$

$$\mathbb{E}[\langle X, \phi_1\rangle \langle X, \phi_2\rangle] = \int_{\mathbb{R}}\int_{\mathbb{R}} K(t,s)\phi_1(t)\phi_2(s)\,dt\,ds,$$

where for $-\infty < s, t < \infty$, $s \neq t$ and $s, t \neq 0$

$$K(t,s) = \log\frac{1}{|t-s|} + g(t,s),$$

where $g(t,s)$ is a continuous function on $\{(t,s) : t > 0, s > 0\}$.

1. The function $g(t,s)$ is given by

$$g(t,s) = \frac{1}{t}\int_0^t \log|s-u|du + \frac{1}{s}\int_0^s \log|t-u|du - \frac{1}{ts}\int_0^t\int_0^s \log|u-v|dudv.$$

2. Recall that the limiting process X has the covariance kernel

$$K(t,s) = \log\frac{1}{|t-s|} + g(t,s),$$

3. Since for $t,s>\delta$ for some $\delta>0$, then $g(t,s)$ is a bounded continuous function. Hence K(t,s) exhibits the same type of singularity as that of a LGF.

Outlines of the proof

1. For $t, s \in \mathbb{R}$, let $K_H(t,s) = \mathbb{E}[X_t^H X_s^H]$.

2. Recall

$$\mathbb{E}[\langle X, \phi_1\rangle \langle X, \phi_2\rangle] = \int_{\mathbb{R}} \int_{\mathbb{R}} K(t,s)\phi_1(t)\phi_2(s)\, dt\, ds.$$

2. Since X^H and X are centered Gaussians taking values in $\mathcal{S}'$, to prove the theorem, it is enough to show that for any $\phi_1, \phi_2 \in \mathcal{S}$,

$$\lim_{H\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} K_H(t,s)\phi_1(t)\phi_2(s)\, ds\, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} K(t,s)\phi_1(t)\phi_2(s)\, ds\, dt.$$

Gaussian Multiplicative Chaos

(Kahane '85), (Rhodes, Vargas '14).

1. Consider a LGF X over a domain D with the covariance kernel
$$K(x,y) = \log_+ \frac{1}{|x-y|} + g(x,y),$$
where g is bounded on $D \times D$.
2. Let $\gamma > 0$. We would like to define
$$M_\gamma(dx) = e^{\gamma X(x)} dx.$$
3. Since X is a distribution this is nontrivial.

Gaussian Multiplicative Chaos

1. Let θ be a smooth modifier, i.e. $\theta \in \mathcal{C}^\infty$, has compact support and $\int_{\mathbb{R}} \theta(x)dx = 1$.
2. Let $\theta_\varepsilon = \frac{1}{\varepsilon}\theta(\cdot\frac{1}{\varepsilon})$ and $X_\varepsilon = X \star \theta_\varepsilon$ the convolution of X and θ_ε.
3. We define random measures

$$M_{\gamma,\varepsilon}(dx) = \exp\Big\{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}E[X_\varepsilon(x)^2]\Big\}dx.$$

4. Then if $\gamma < \sqrt{2}$, $M_{\gamma,\varepsilon}$ converge in probability in the space of Radon measures to M_γ (topology of weak convergence).

Gaussian Multiplicative Chaos -Convergence

The limiting measure M_γ is called Gaussian Multiplicative Chaos.

1. From Fubini we have for any compact A,

$$E[M_{\gamma,\varepsilon}(A)] = \int_A E\Big[e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} E[X_\varepsilon(x)^2]}\Big] dx = |A|.$$

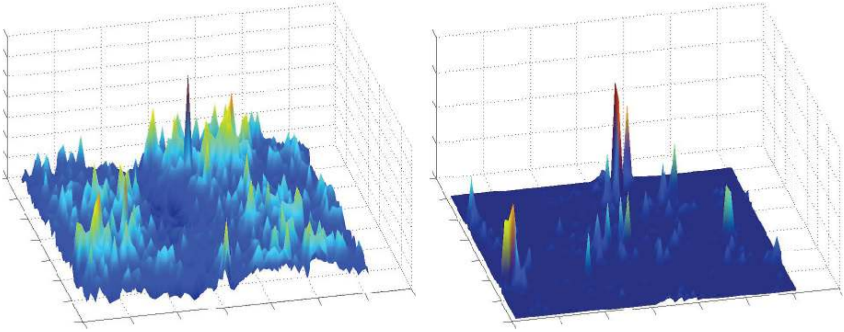
2. This explains the normalization term $\frac{\gamma^2}{2} E[X_\varepsilon(x)^2]$.

Gaussian Multiplicative Chaos -Convergence

1. From Fubini and dominated convergence,

$$
\begin{aligned}
E[M_{\gamma,\varepsilon}(A)^2] &= E\Big[\Big(\int_A e^{\gamma X_\varepsilon(x)-\frac{\gamma^2}{2}E[X_\varepsilon(x)^2]}dx\Big)^2\Big] \\
&= E\Big[\int_A\int_A e^{\gamma X_\varepsilon(x)+\gamma X_\varepsilon(y)-\frac{\gamma^2}{2}E[X_\varepsilon(x)^2]-\frac{\gamma^2}{2}E[X_\varepsilon(y)^2]}dxdy\Big] \\
&= \int_A\int_A e^{\gamma^2 E[X_\varepsilon(x)X_\varepsilon(y)]}dxdy \\
&\to \int_A\int_A e^{\gamma^2 K(x,y)}dxdy.
\end{aligned}
$$

2. Use this to show $E[\big(M_{\gamma,\varepsilon}(A)-M_{\gamma,\varepsilon'}(A)\big)^2]\to 0$, as $\varepsilon,\varepsilon'\to 0$.



(a) $\gamma = 0.2$ (b) $\gamma = 1$

Realizations of GMC for different values of γ (by Rhodes, Vargas '14)

Geometric fBm with $H = 0$

1. Motivated by the previous result, we first fix $\delta > 0$ and define an approximate volatility measure ξ_γ^H by

$$\xi_\gamma^H(dt) = e^{\gamma X_t^H - \frac{\gamma^2}{2}\mathbb{E}[(X_t^H)^2]}dt, \quad \delta \le t \le 1,$$

   for some constant $\gamma > 0$. Here we assume that $\xi_\gamma^H(\cdot)$ vanishes on $[\delta, 1]^c$.

2. In what follows, convergence in the L^1 norm stands for the usual convergence of random variables in L^1.

Geometric fBm with $H = 0$

Theorem (N. and Rosenbaum, 2018)

For $\gamma < \sqrt{2}$, $\{\xi_\gamma^H\}_{H \in (0,1)}$ converges as H approaches zero to a random measure ξ_γ in the following sense,

$$\int_{\mathbb{R}} \phi(t) \xi_\gamma^H(dt) \xrightarrow{L^1} \int_{\mathbb{R}} \phi(t) \xi_\gamma(dt), \quad \text{for all } \phi \in \mathcal{S}.$$

Moreover, the limiting measure ξ_γ is Gaussian multiplicative chaos.

Multifractal Analysis

Multifractal Analysis is the study of sets S_h where a function f has a given Hölder exponent h.

- Determination of $d(h)$ - the Hausdorff dimension of S_h.
- The function $d(h)$ is called the *spectrum of singularities* of f.

Multifractal Analysis

- Pointwise Regularity -P_{t_0} polynomial of degree at most $\lfloor l \rfloor$ and

$$|f(t) - P_{t_0}(t)| \leq C|t - t_0|^l.$$

- Hölder exponent of f at t_0

$$h_f(t_0) = \sup\{l : f \in C^l(t_0)\}.$$

Multifractal Analysis - examples

1. $X_t = C \cdot t$ then $d(h) = -\infty \ \ \forall h$.
2. X_t compound Poisson Process with drift. The number of jumps is finite. $d(0) = 0$ and $d(h) = -\infty$ else.
3. X_t is a Brownian motion then $d(1/2) = 1$ and $d(h) = -\infty$ else (Orey, Taylor 1979).
4. X_t is a superposition of Brownian motion and compound Poisson, then $d(0) = 0, d(1/2) = 1$ and $d(h) = -\infty$ else.

Multifractal Analysis - Lévy Processes

- Let $X = \{X(t)\}_{t>0}$ be a *Lévy process* with a *Lévy measure* π.
- When $\pi(\mathbb{R}) = \infty$ the growth of the Lévy measure near the origin can be estimated by

$$\beta = \inf\left\{\gamma \geq 0 : \int_{|x|\leq 1} |x|^{\gamma}\pi(dx) < \infty\right\}$$

- Since $\pi(x)$ is a Lévy measure, therefore $0 \leq \beta \leq 2$.

Spectrum of Singularities - Lévy processes

Theorem (Jaffard, 1999)

$X(t)$ has no Brownian component the spectrum of singularities of almost every sample path of $X(t)$ is:

$$d_\beta(h) = \begin{cases} \beta h & \text{if } h \in [0, 1/\beta], \\ \\ -\infty & \text{otherwise;} \end{cases}$$

Frisch-Parisi conjecture

1. For a real valued function f define

$$S_p(\ell) = \int |f(x+\ell) - f(x)| dx.$$

2. Suppose that $S_p(\ell)$ scales like $|\ell|^{\zeta_f(p)}$, when $\ell \to 0$.

3. **Multifractal Formalism:** Frisch and Parisi (1985) conjectured that

$$d_f(h) = \inf_p \{h \cdot p - \zeta_f(p) + 1\}.$$

Multifractal Properties

We describe the behavior of the moments of ξ_γ (Vargas, Rhodes '16).
Let $B(t,r)$ be ball of radius r, centred at t.

For all $t \in (\delta, 1)$ and $q \in (-\infty, 2/\gamma^2)$, there exists $C(t,q) > 0$ such that

$$\mathbb{E}\big[\xi_\gamma\big(B(t,r)\big)^q\big] \sim C(t,q) r^{\zeta(q)}, \quad \text{as } \to 0,$$

where

$$\zeta(q) = (1+\gamma^2/2)q - \gamma^2 q^2/2.$$

Multifractal Properties

1. Next we describe the spectrum of singularities of ξ_γ.

2. For any $0 < \gamma < \sqrt{2}$ and $0 < r < \sqrt{2}/\gamma$, we define

$$G_{\gamma,r} = \Big\{x \in (\delta, 1);\ \lim_{\varepsilon \to 0} \frac{\log \xi_\gamma(B(x,\varepsilon))}{\log \varepsilon} = 1 + (\frac{1}{2} - r)\gamma^2\Big\}.$$

3. The set $G_{\gamma,r}$ somehow corresponds to the points x where the Hölder regularity of ξ_γ is equal to $1 + (1/2 - r)\gamma^2$.

Let $dim_H(A)$ denote the Hausdorff dimension of a set A. Then we have

$$dim_H(G_{\gamma,r}) = 1 - \frac{\gamma^2 r^2}{2}.$$

In particular, we remark that

$$dim_H(G_{\gamma,r}) = \inf_{p \in \mathbb{R}} \Big\{ p(1 + (\frac{1}{2} - r)\gamma^2) - \zeta(p) + 1 \Big\}.$$

This equality means that the Frish-Parisi (1985) conjecture relating the scaling exponents of a process to its spectrum of singularities holds in our case.

Open problem: asset price behaviour for $H = 0$

1. Consider a rough volatility model (e.g. Heston):

$$
\begin{aligned}
dS_t &= S_t \sqrt{V_t} dW_t \\
dV_t &= \lambda(\theta - V_t)dt + \nu \sqrt{V_t} dB_t^H,
\end{aligned}
$$

   where B^H and W are negatively correlated.

2. Make sense of the price $S = S(H)$ as $H \downarrow 0$, and derive its properties.


THANK YOU
for your
ATTENTION!