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Uniqueness, existence and regularity of stochastic Volterra integral equations

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Kernel functions and path-dependency

Let us regard the following stochastic Volterra integral equation with an initial value condition:

$$X_t = X_0 + \int_0^t K(t,s)\overline{b}(s, X_s)\, ds + \int_0^t K(t,s)\overline{\sigma}(s, X_s)\, dW_s \quad (1)$$

for $t \in [0, T]$ a.s. In this one-dimensional setting, $K : [0, T]^2 \to \mathbb{R}$ denotes the kernel function and

$$\overline{b}, \overline{\sigma} : [0, T] \times \mathbb{R} \to \mathbb{R}$$

are measurable. For instance, one can take $K(t,s) = (t-s)^{\beta-1}$ for all $s, t \in [0, T]$ with $s < t$ and some $\beta > 0$.

The case of regular kernels:

(i) Such equations were first analyzed by Berger and Mizel (1980), even in a multidimensional setting.

(ii) Protter (1985) showed the existence and uniqueness of strong solutions, allowing for path-dependency and a semimartingale as integrator.

The case of irregular kernels:

(iii) Zhang (2010) proved the existence and uniqueness of strong solutions with continuous paths, even in a Banach space.

(iv) Abi Jaber, Larsson and Pulido (2017) considered weak solutions if the coefficients are affine.

(i) In the sequel, let $C_r^\alpha([0,T],E)$ denote the Banach space of all $x \in C([0,T],E)$ that are α-Hölder-continuous on $[r,T]$, endowed with the 'delayed α-Hölder norm'

$$\|x\|_{\alpha,r} := \|x^r\|_\infty + \sup_{s,t\in[r,T]:\, s\neq t} \frac{|x(s)-x(t)|}{|s-t|^\alpha}.$$

(ii) Let $\mathscr{L}^{\alpha,q}([r,T]^2,E)$ be the linear space of all measurable maps $K : [r,T]^2 \to E$ for which there is $c \geq 0$ such that

$$\left(\int_s^t |K(t,s')|^q\, ds'\right)^{1/q} + \left(\int_r^s |K(t,r') - K(s,r')|^q\, dr'\right)^{1/q}$$

is bounded by $c(t-s)^\alpha$ for every $s,t \in [r,T]$ with $s < t$.

Example

Let E be a Banach algebra, $k \in C^\alpha([r,T],E)$ and $l : [r,T] \to E$ be measurable. Then $K : [r,T]^2 \to E$ given by

$$K(t,s) := k(t)l(s)$$

belongs to $\mathscr{L}^{\alpha,q}([r,T]^2,E)$ as soon as there is $\tilde{q} > q$ such that

$$\int_r^T |l(t)|^{\tilde{q}}\,dt < \infty \quad \text{and} \quad \alpha \leq 1/q - 1/\tilde{q}.$$

Deterministic Volterra integral equations

First, we consider the following path-dependent Volterra integral equation coupled with a running value condition:

$$x(t) = \hat{x}(t) + \int_r^t K(t,s)\varphi(s, x^s)\, ds \tag{v}$$

for $t \in [r, T]$ and $x(q) = \hat{x}(q)$ for $q \in [0, r]$. In this framework, D is an open set in the separable Banach space E and

$$K : [r, T]^2 \to \mathbb{R} \quad \text{and} \quad \varphi : [r, T] \times C([0, T], D) \to E$$

are two measurable maps.

We call an interval I in $[0, T]$ admissible if it agrees with either $[0, t]$ or $[0, t)$ for some $t \in (r, T]$.

Definition

A solution to (v) on an admissible interval I is a map $x \in C(I, D)$ such that $x^r = \hat{x}^r$, as well as $\int_r^t |K(t,s)\varphi(s, x^s)|\, ds < \infty$ and

$$x(t) = \hat{x}(t) + \int_r^t K(t,s)\varphi(s, x^s)\, ds$$

for all $t \in I$ with $t \geq r$.

We let K satisfy a Hölder continuity condition and φ a boundedness and Lipschitz condition on a class of bounded sets.

(v.1) There are $\alpha \in (0,1]$ and $q > 1$ so that $K \in \mathscr{L}^{\alpha,q}([r,T]^2, \mathbb{R})$.

(v.2) For any closed and bounded set B in D that is bounded away from ∂D there are $k, \lambda \in \mathscr{L}^p([r,T], \mathbb{R}_+)$ such that

$$|\varphi(s,x)| \leq k(s), \quad |\varphi(s,x) - \varphi(s,y)| \leq \lambda(s)\|x-y\|_\infty$$

for any $s \in [r,T]$ and $x, y \in C([0,T], B)$, where p is the dual exponent of q.

Unique non-extendible solutions (K., 2019)

Under (v.1) and (v.2), there is a unique non-extendible solution $x_{r,\hat{x}}$ to (v) on a maximal interval of existence $I_{r,\hat{x}}$ that is open in $[0, T]$. For $t^+_{r,\hat{x}} := \sup I_{r,\hat{x}}$ we either have $I_{r,\hat{x}} = [0, T]$ or

$$\lim_{t \uparrow t^+_{r,\hat{x}}} \min \left\{ \operatorname{dist}(x_{r,\hat{x}}(t), \partial D), \frac{1}{1 + |x_{r,\hat{x}}(t)|} \right\} = 0. \qquad (2)$$

Example

Assume that

(i) there is $\beta \in (0, 1]$ such that

$$K(t, s) = (t - s)^{\beta - 1}$$

for all $s, t \in [r, T]$ with $s < t$, and

(ii) there is a map $\psi : D \to E$ that is Lipschitz continuous on any bounded set in D that is bounded away from ∂D with

$$\varphi(s, x) = \psi(x(s))$$

for any $s \in [r, T]$ and $x \in C([0, T], D)$.

Then (v.1) and (v.2) are valid, and the proposition applies.

Next, we require that $\|\hat{x}\|_{\alpha,r} < \infty$ and add an affine boundedness condition on φ.

(v.3) There are $k, l \in \mathscr{L}^p([r, T], \mathbb{R}_+)$ so that

$$|\varphi(s, x)| \leq k(s) + l(s)\|x\|_\infty$$

for all $s \in [r, T]$ and $x \in C([0, T], D)$.

From (v.3) it follows that any global solution to (v) lies in the set $\mathscr{R}^{\alpha,p}_{r,\hat{x}}$ of all $x \in C^\alpha_r([0,T],E)$ satisfying

$$\frac{\|x^t - \hat{x}^t\|^p_{\alpha,r}}{r_p} \leq e^{r_p \int_r^t l(s)^p \, ds} \left(\int_r^t k(s)^p + l(s)^p \|\hat{x}^s\|^p_\infty \, ds \right)$$

for each $t \in [r,T]$ and some constant $r_p \geq 0$.

Global delayed Hölder continuous solutions

Let (v.1)-(v.3) hold and $D = E$. Then $I_{r,\hat{x}} = [0, T]$ and for any $x_0 \in \mathscr{R}^{\alpha,p}_{r,\hat{x}}$ the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{R}^{\alpha,p}_{r,\hat{x}}$, recursively given by

$$x_{n+1}(t) := \hat{x}(t) + \int_r^{r\vee t} K(t,s)\varphi(s, x_n^s)\, ds$$

for all $n \in \mathbb{N}_0$, converges in the delayed α-Hölder norm $\|\cdot\|_{\alpha,r}$ to $x_{r,\hat{x}}$, the unique global solution to (v).

Estimates in the delayed Hölder norm

Moreover, the Picard sequence $(x_n)_{n\in\mathbb{N}}$ satisfies

$$\|x_n^t - \hat{x}^t\|_{\alpha,r}^p \leq r_p \sum_{i=0}^{n-1} \frac{L_p(t)^i}{i!} \left(\int_r^t k(s)^p + l(s)^p \|\hat{x}^s\|_\infty^p \, ds \right) + \frac{L_p(t)^n}{n!} \|x_0^t - \hat{x}^t\|_\infty^p,$$

where $L_p(t) := r_p \int_r^t l(s)^p \, ds$, and

$$\|x_n^t - x_{r,\hat{x}}^t\|_{\alpha,r} \leq \|x_1^t - x_0^t\|_\infty \sum_{i=n}^{\infty} \left(\frac{1}{i!} \right)^{1/p} \left(r_p \int_r^t \lambda(s)^p \, ds \right)^{i/p}$$

for all $n \in \mathbb{N}$, every $t \in [r, T]$ and some $\lambda \in \mathscr{L}^p([r, T], \mathbb{R}_+)$.

Example

Suppose that $D = E$,

(i) there exists $\beta \in (0, 1]$ with $\hat{x} \in C_r^{\beta-}([0, T], E)$ and $K(t, s) = (t - s)^{\beta-1}$ for any $s, t \in [r, T]$ with $s < t$, and

(ii) there is a map $\psi : E \to E$ that is Lipschitz continuous on bounded sets and of linear growth such that $\varphi(s, x) = \psi(x(s))$ for each $s \in [r, T]$ and $x \in C([0, T], E)$.

Then $x_{r,\hat{x}} \in C_r^{\beta-}([0, T], E)$ and the Picard sequence converges in the delayed α-Hölder norm $\|\cdot\|_{\alpha,r}$ to $x_{r,\hat{x}}$ for each $\alpha \in [0, \beta)$.

Continuity of the solution map

Let $D = E$ and (v.1)-(v.3) be satisfied. Then the map

$$C_r^\alpha([0,T],E) \to C_r^\alpha([0,T],E), \quad \hat{x} \mapsto x_{r,\hat{x}}$$

is Lipschitz continuous on bounded sets. That is, for any $n \in \mathbb{N}$ there is $\lambda_n \geq 0$ such that

$$\|x_{r,\hat{x}} - x_{r,\hat{y}}\|_{\alpha,r} \leq \lambda_n \|\hat{x} - \hat{y}\|_{\alpha,r}$$

for each $\hat{x}, \hat{y} \in C_r^\alpha([0,T],E)$ with $\|\hat{x}\|_{\alpha,r} \vee \|\hat{y}\|_{\alpha,r} \leq n$.

Stochastic Volterra integral equations

We turn to the following path-dependent stochastic Volterra integral equation combined with a running value condition:

$$X_t = \xi_t + \int_r^t K(t,s) b(s, X^s)\, ds + \int_r^t K(t,s)\sigma(s, X^s)\, dW_s \quad \text{(V)}$$

for $t \in [r, T]$ and $X^r = \xi^r$ a.s. Here, H and L are two separable Hilbert spaces, $D \subset H$ is open, W is an id_L-cylindrical Wiener process and

$$b : [r, T] \times C([0, T], D) \to H,$$
$$\sigma : [r, T] \times C([0, T], D) \to \mathscr{L}(L, H)$$

are two measurable maps.

Definition

A strong solution to (V) is a pair (X, τ) consisting of a stopping time τ satisfying $\tau > r$ and some adapted continuous process $X : \{(t, \omega) \in [0, T] \times \Omega \,|\, t < \tau(\omega)\} \to H$ such that

$$\int_r^t |K(t,s)b(s, X^s)|\, ds + \int_r^t |K(t,s)\sigma(s, X^s)|^2\, ds < \infty$$

and (V) hold a.s. on $\{t < \tau\}$ for each $t \in [0, T]$.

Let K satisfy a Hölder continuity condition and (b, σ) a boundedness and Lipschitz condition on certain bounded sets.

(V.1) There are $\alpha \in (0, 1]$ and $q > 2$ with $K \in \mathscr{L}^{\alpha,q}([r, T]^2, \mathbb{R})$.

(V.2) For any closed and bounded set B in D that is bounded away from ∂D there are $k, \lambda \in \mathscr{L}^p([r, T], \mathbb{R}_+)$ so that

$$|b(s,x)| \vee |\sigma(s,x)| \leq k(s),$$
$$|b(s,x) - b(s,y)| \vee |\sigma(s,x) - \sigma(s,y)| \leq \lambda(s)\|x - y\|_\infty$$

for every $s \in [r, T]$ and $x, y \in C([0, T], B)$, where p is the dual exponent of $q/2$.

Unique non-extendible strong solutions (K., 2019)

Under (V.1) and (V.2), there is a unique strong solution $(X^{r,\xi}, \tau_{r,\xi})$ to (V) that is non-extendible in the sense that

$$\lim_{t \uparrow \tau_{r,\xi}} \min \left\{ \operatorname{dist}(X_t^{r,\xi}, \partial D), \frac{1}{1 + |X_t^{r,\xi}|} \right\} = 0 \tag{3}$$

a.s. on $\{\tau_{r,\xi} < \infty\}$.

First, we see that the linear space $\mathscr{C}_r^{\alpha,p}([0,T],H)$ of all adapted continuous processes $X : [0,T] \times \Omega \to H$ satisfying $\mathbb{E}[\|X\|_{\alpha,r}^p] < \infty$ is a complete seminormed space.

Convergence in $\|\cdot\|_{\alpha,r}$ in p-th moment

A sequence $({}_nX)_{n\in\mathbb{N}}$ in $\mathscr{C}_r^{\alpha,p}([0,T],H)$ converges if and only if $(\|{}_nX\|_{\alpha,r})_{n\in\mathbb{N}}$ is p-fold uniformly integrable and

$$\lim_{n\uparrow\infty} \mathbb{P}(\|{}_nX - X\|_{\alpha,r} \geq \varepsilon) = 0 \quad \text{for all } \varepsilon > 0$$

and an adapted continuous process $X : [0,T] \times \Omega \to H$. In the latter case, $\mathbb{E}[\|X\|_{\alpha,r}^p] < \infty$ and $\lim_{n\uparrow\infty} \mathbb{E}[\|{}_nX - X\|_{\alpha,r}^p] = 0$.

Now let $\mathbb{E}\big[\|\xi\|_{\alpha,r}^p\big] < \infty$ and consider an affine boundedness and a Lipschitz condition on (b, σ).

(V.3) There are $k, l \in \mathscr{L}^p([r, T], \mathbb{R}_+)$ such that

$$|b(s, x)| \vee |\sigma(s, x)| \leq k(s) + l(s)\|x\|_\infty$$

for any $s \in [r, T]$ and $x \in C([0, T], D)$.

(V.4) There is $\lambda \in \mathscr{L}^p([r, T], \mathbb{R}_+)$ such that

$$|b(s, x) - b(s, y)| \vee |\sigma(s, x) - \sigma(s, y)| \leq \lambda(s)\|x - y\|_\infty$$

for every $s \in [r, T]$ and $x, y \in C([0, T], D)$.

Condition (V.3) implies that any global solution to (V) belongs to the set $\mathscr{C}^{\alpha,p}_{r,\xi}$ of all $X \in \mathscr{C}^{\alpha,p}_r([0,T],H)$ such that

$$\frac{\mathbb{E}\left[\|X^t - \xi^t\|^p_{\alpha,r}\right]}{c_{\alpha,p}} \leq e^{c_{\alpha,p}\int_r^t l(s)^p\,ds}\left(\int_r^t k(s)^p + l(s)^p \mathbb{E}\left[\|\xi^s\|^p_\infty\right] ds\right)$$

for any $t \in [r,T]$ and some constant $c_{\alpha,p} \geq 0$.

Global delayed Hölder continuous strong solutions

Let (V.1)-(V.4) be valid and $D = H$. Then $\tau_{r,\xi} = \infty$ and for any ${}_0X \in \mathscr{C}^{\alpha,p}_{r,\xi}$ the sequence $({}_nX)_{n\in\mathbb{N}}$ in $\mathscr{C}^{\alpha,p}_{r,\xi}$, recursively given by

$$
\begin{aligned}
{}_{n+1}X_t = \xi_t &+ \int_r^{r\vee t} K(t,s) b(s, {}_nX^s)\, ds \\
&+ \int_r^{r\vee t} K(t,s) \sigma(s, {}_nX^s)\, dW_s \quad \text{a.s.}
\end{aligned}
$$

for all $n \in \mathbb{N}_0$, converges in the delayed α-Hölder norm $\|\cdot\|_{\alpha,r}$ in p-th mean to $X^{r,\xi}$, the unique global solution to (V). That is,

$$
\lim_{n\uparrow\infty} \mathbb{E}\big[\|{}_nX - X^{r,\xi}\|^p_{\alpha,r}\big] = 0.
$$

Estimates in the delayed Hölder norm in p-th mean

Further, for any $n \in \mathbb{N}$ and each $t \in [r, T]$, $\mathbb{E}\big[\|{}_nX^t - \xi^t\|^p_{\alpha,r}\big]$ is bounded by

$$c_{\alpha,p} \sum_{i=0}^{n-1} \frac{L_{\alpha,p}(t)^i}{i!} \left(\int_r^t k(s)^p + l(s)^p \mathbb{E}\big[\|\xi^s\|^p_\infty\big]\, ds \right) + \frac{L_{\alpha,p}(t)^n}{n!} \mathbb{E}\big[\|{}_0X^t - \xi^t\|^p_\infty\big],$$

where $L_{\alpha,p}(t) := c_{\alpha,p} \int_r^t l(s)^p\, ds$, and $\big(\mathbb{E}\big[\|{}_nX_{\cdot\wedge t} - X^{r,\xi}_{\cdot\wedge t}\|^p_{\alpha,r}\big]\big)^{1/p}$ cannot exceed

$$\big(\mathbb{E}\big[\|{}_1X^t - {}_0X^t\|^p_\infty\big]\big)^{1/p} \sum_{i=n}^{\infty} \left(\frac{1}{i!}\right)^{1/p} \left(c_{\alpha,p} \int_r^t \lambda(s)^p\, ds \right)^{i/p}.$$

Example

Assume that $D = H$,

(i) there is $\beta \in (0, 1/2]$ with $\xi \in \mathscr{C}_r^{\beta-,p}([0,T],H)$ for any $p \geq 1$ and $K(t,s) = (t-s)^{\beta-1/2}$ for all $s,t \in [r,T]$ with $s < t$, and

(ii) there are two maps $\varphi : H \to H$ and $\psi : H \to \mathscr{L}(L,H)$ that are Lipschitz continuous on bounded sets and of linear growth such that for each $s \in [r,T]$ and $x \in C([0,T],H)$,

$$b(s,x) = \varphi(x(s)) \quad \text{and} \quad \sigma(s,x) = \psi(x(s)).$$

Then $X^{r,\xi} \in \mathscr{C}_r^{\beta-,p}([0,T],H)$ and the Picard sequence converges in the norm $\|\cdot\|_{\alpha,r}$ in p-th moment to $X^{r,\xi}$ for each $p \geq 1$ and $\alpha \in [0,\beta)$.

Continuity of the strong solution map

Let $D = H$ and (V.1)-(V.4) be valid. Then the map

$$\mathscr{C}_r^{\alpha,p}([0,T],H) \mapsto \mathscr{C}_r^{\alpha,p}([0,T],H), \quad \xi \mapsto X^{r,\xi}$$

is Lipschitz continuous. Put differently, there is $\lambda \geq 0$ satisfying

$$\left(\mathbb{E}\left[\|X^{r,\xi} - X^{r,\eta}\|_{\alpha,r}^p\right]\right)^{1/p} \leq \lambda\left(\mathbb{E}[\|\xi - \eta\|_{\alpha,r}^p]\right)^{1/p}$$

for any $\xi, \eta \in \mathscr{C}_r^{\alpha,p}([0,T],H)$.

Thank you for your attention!

References

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