Liquidity and Asset Prrices

Johannes Muhle-Karbe

Imperial College London

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Outline

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Introduction

Liquidity and Asset Prices

- Interplay of *liquidity* and *asset prices*.
  - Higher returns for less liquid assets?
  - How does a transaction tax affect the volatility of a financial market?
- Such questions need to be studied with *equilibrium* models.
  - Prices determined as output by matching supply and demand, rather than modeled as input.
- Equilibrium analyses are generally hard. Fixed points.
- Intractability is compounded with frictions.
  - *"The problem is that we don't have enough math. Frictions are just hard with the tools we have right now."* (Cochrane '10).

Introduction

Literature

Extant literature focused on restrictive settings.

- Numerical solution of simple tree models:
  - Heaton/Lucas ‘96. Buss/Dumas ‘15; Buss/Vilkov/Uppal ‘15.
- Deterministic asset prices (no volatility):
  - Vayanos/Villa ‘99; Lo/Mamaysky/Wang ‘04; Weston ‘17.
- Assets with exogenous volatilities:
  - Vayanos ‘98; Garleanu/Pedersen ‘16; Sannikov/Skrzypacz ‘16; Bouchard/Fukasawa/Herdegen/M-K ‘18.
- Deterministic trading strategies (Vayanos ‘98).

- Models with realistic dynamics for prices and trading volume?
- Link between trading costs, asset returns, and volatilities?
- Do the effects matter for realistic parameter values?

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Risk-Sharing Economy

- Exogenous savings account. Price normalized to one.
- Fixed supply $s > 0$ of risky asset with Itô dynamics:

$$dS_t = \mu_t dt + \sigma_t dW_t$$

- Initial price S_0, expected returns $(\mu_t)_{t\in[0,T]}$ and volatility $(\sigma_t)_{t\in[0,T]}$ to be determined in equilibrium.
- Agents $n = 1, 2$ trade to hedge fluctuations of their endowments

$$dY_t^n = \beta_t^n dW_t$$

- Frictionless wealth dynamics of a trading strategy $(\varphi_t)_{t\in[0,T]}$:

$$\varphi_t dS_t + dY_t^n$$

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Goal Functionals and Equilibrium

- Simplest mean-variance goal functional:

$$E\left[\int_0^T(\varphi_t dS_t + dY_t^n) - \frac{\gamma^n}{2}\int_0^T \langle \varphi_t dS_t + dY_t^n \rangle\right]$$
$$= E\left[\int_0^T \varphi_t \mu_t - \frac{\gamma^n}{2}(\varphi_t \sigma_t + \beta_t^n)^2 dt\right] \to \max!$$

- Optimum directly given by pointwise maximization:

$$\varphi_t^n = \frac{\mu_t}{\gamma^n \sigma_t^2} - \frac{\beta_t^n}{\sigma_t}$$

- Supply s in turn determines equilibrium return:

$$\mu_t = \bar{\gamma}\sigma_t^2\left(s + \frac{\beta_t^1}{\sigma_t} + \frac{\beta_t^2}{\sigma_t}\right), \quad \text{where } \bar{\gamma} = \frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}$$

- What about the equilibrium volatility?

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Equilibrium ct'd

- Supply matches demand for *any* volatility $(\sigma_t)_{t\in[0,T]}$ and

$$\mu_t = \bar{\gamma}\sigma_t^2 \left(s + \frac{\beta_t^1}{\sigma_t} + \frac{\beta_t^2}{\sigma_t} \right)$$

- Simplest way to pin down volatility: exogenous terminal condition,

$$S_T = \mathfrak{S}$$

- Fundamental value or (expectation of) future dividends.
- Equilibrium price is in turn determined by a (quadratic) "Backward Stochastic Differential Equation" (BSDE):

$$dS_t = \left(\bar{\gamma} s \sigma_t^2 + \bar{\gamma}\sigma_t(\beta_t^1 + \beta_t^2) \right) dt + \sigma_t dW_t, \quad S_T = \mathfrak{S}$$

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Equilibrium ct'd

- Volatility is now part of the solution.
  - Needs to be chosen appropriately to steer (adapted) solution into terminal condition.
- For bounded inputs $\mathfrak{S}$, β^1, β^2, existence and uniqueness of scalar quadratic BSDEs is standard.
- Here: explicit solution in terms of Laplace transform of $\mathfrak{S}$:

$$S_t = -\frac{1}{2\bar{\gamma}} \log E_t^\beta \left[e^{-2\bar{\gamma}\mathfrak{S}} \right] \quad \text{where } \frac{d\mathbb{P}^\beta}{d\mathbb{P}} = \mathcal{E}\left(-\bar{\gamma} \int_0^\cdot (\beta_t^1 + \beta_t^2) dt \right)$$

- Example: if $\beta^1 + \beta^2 = 0$ and $\mathfrak{S} = bT + aW_T$, then equilibrium price has Bachelier dynamics:

$$S_t = (b - \bar{\gamma} s a^2 T) + \bar{\gamma} s a^2 t + a W_t$$

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Goal Functional with Transaction Costs

- Model with *quadratic* trading costs,

$$J(\dot{\varphi}) = E\left[\int_0^T \left(\varphi_t\mu_t - \frac{\gamma^n}{2}(\varphi_t\sigma_t + \beta_t^n)^2 - \frac{\lambda}{2}\dot{\varphi}_t^2\right)dt\right] \to \max!$$

- Standard in market microstructure (Almgren/Chriss '01).
  - Penalty on size and speed of adjustments.
  - Most tractable specification due to linear first-order condition. (Garleanu/Pedersen '13/'16, Cartea/Jaimungal '16, Bank/Soner/Voss '17).
- Equilibrium dynamics?
  - First solve individual problems for fixed μ, σ.
  - Then pin down μ, σ by matching supply and demand as well as the terminal condition.
- With trading costs, optimization is no longer pointwise.
  - Current position becomes extra state variable.

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Individual Optimality

- Fix return $(\mu_t)_{t\in[0,T]}$ and volatility $(\sigma_t)_{t\in[0,T]}$.
- Necessary and sufficient for optimality:
  - Directional derivative $\lim_{\rho\to 0}\frac{1}{\rho}(J(\dot\varphi+\rho\dot\psi)-J(\dot\varphi))$ vanishes for *any* perturbation ψ:

$$0 = E_t\left[\int_0^T \left(\mu_t \int_0^t \dot\psi_u du - \gamma^n \sigma_t(\varphi_t\sigma_t+\beta_t^n)\int_0^t \dot\psi_u du - \lambda\dot\varphi_t\dot\psi_t\right)dt\right]$$

  - Rewrite using Fubini's theorem:

$$0 = E_t\left[\int_0^T \left(\int_t^T \left(\mu_u - \gamma^n\sigma_u(\varphi_u\sigma_u+\beta_u^n)\right)du - \lambda\dot\varphi_t^\top\right)\dot\psi_t dt\right]$$

  - Has to hold for *any* perturbation $\dot\psi_t$.

- Whence, tower property of conditional expectation yields:

$$\dot{\varphi}_t = \frac{1}{\lambda} E_t \left[\int_t^T \Big(\mu_u - \gamma^n \sigma_u \left(\varphi_u \sigma_u + \beta_u^n \right) \Big) du \right]$$
$$= M_t - \frac{1}{\lambda} \int_0^t \Big(\mu_u - \gamma^n \sigma_u \left(\varphi_u \sigma_u + \beta_u^n \right) \Big) du$$

  for a martingale M_t.

- Optimal strategy solves a Forward-Backward SDE (FBSDE):

$$d\varphi_t^n = \dot{\varphi}_t^n dt, \qquad \varphi_0^n = \text{initial position}$$
$$d\dot{\varphi}_t^n = dM_t + \frac{\gamma^n}{\lambda} \Big(\sigma_u^2 \varphi_t^n - \frac{\mu_t}{\gamma^n} + \sigma_u \beta_u^n \Big) dt, \quad \dot{\varphi}_T^n = 0$$

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Individual Optimality and FBSDEs ct'd

- If the volatility $\sigma_t \equiv \sigma$ is *constant*:
  - FBSDE for $(\varphi^1, \dot{\varphi}^1)$ can be reduced to a scalar Riccati ODE by a suitable ansatz.
  - Explicit solution in terms of conditional expectations of inputs μ, β^1, β^2.
- If the volatility $(\sigma_t)_{t \in [0,T]}$ is stochastic (and bounded):
  - ODE replaced by backward *stochastic* Riccati equation.
  - Still a *scalar* equation.
  - Existence and uniqueness established by Kohlmann/Tang '02 using comparison arguments.
  - In turn allows to describe $(\varphi^1, \dot{\varphi}^1)$ using conditional expectations.
- Equilibrium?

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Market Clearing

- To match supply and demand, need

$$\begin{aligned}0 &= d\dot{\varphi}^1_t + d\dot{\varphi}^2_t \\ &= \left(\tfrac{\sigma_t^2}{\lambda}(\gamma^1\varphi^1_t + \gamma^2\varphi^2_t) + \tfrac{\sigma_t}{\lambda}(\gamma^1\beta^1_t + \gamma^2\beta^2_t) - \tfrac{2\mu_t}{\lambda}\right)dt + dM_t\end{aligned}$$

- In equilibrium: $\varphi^2_t = s - \varphi^1_t$.
- Equilibrium return therefore has to satisfy

$$\mu_t = \sigma_t \tfrac{\gamma^1\beta^1_t + \gamma^2\beta^2_t}{2} + \sigma_t^2 \tfrac{\gamma^2 s}{2} + \sigma_t^2 \tfrac{\gamma^1 - \gamma^2}{2}\varphi^1_t$$

- Plug back into FBSDE corresponding to agent 1's optimality condition $\rightsquigarrow$ FBSDE for *equilibrium* strategy of agent 1.

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Equilibrium and FBSDEs

- Equilibrium position φ^1_t and trading rate $\dot\varphi_t$ have to solve

$$d\varphi^1_t = \dot\varphi^1_t dt, \quad \varphi^1_0 = \text{initial position}$$
$$d\dot\varphi^1_t = \frac{1}{\lambda}\left(\sigma_t \frac{\gamma^1\beta^1_t - \gamma^2\beta^2_t}{2} - \sigma^2_t\frac{\gamma^2 s}{2} + \sigma^2_t\frac{\gamma^1+\gamma^2}{2}\varphi^1_t\right)dt + dM^1_t, \quad \dot\varphi^1_T = 0$$

- What about the equilibrium volatility $(\sigma_t)_{t\in[0,T]}$?
- As without frictions, has to match terminal condition:

$$dS_t = \left[\sigma_t\frac{\gamma^1\beta^1_t+\gamma^2\beta^2_t}{2} + \sigma^2_t\frac{\gamma^2 s}{2} + \sigma^2_t\frac{\gamma^1-\gamma^2}{2}\varphi^1_t\right]dt + \sigma_t dW_t, \quad S_T = \mathfrak{S}$$

- In summary: equilibrium price (S,σ), position φ^1, trading rate $\dot\varphi^1$ solve fully coupled system of FBSDEs.

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Well-posedness for FBSDEs?

- No general theory.
  - Well-posedness can fail even for linear systems.
  - Positive results need to be established on a case-by-case basis.
- The system arising here:
  - Has a multidimensional backward component.
  - Is not Lipschitz and generally not even of quadratic growth.
  - Is fully coupled.
- Existence and Uniqueness? Properties fo the solution?
- Picard iteration only works if time horizon T is short enough.
- But BSDE for equilibrium price decouples for $\gamma^1 = \gamma^2$.
  - Equilibrium price S and volatility σ then coincide with frictionless counterparts $\bar{S}$, $\bar{\sigma}$.
  - Expansion around this case for $\gamma^1 \approx \gamma^2$?

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Similar Risk Aversions

- Herdegen/Possamai/M-K '19:
  - FBSDE for equilibrium price has a solution given that $|\gamma^1 - \gamma^2|$ sufficiently small.
  - Provides unique frictional equilibrium in a neighbourhood of it's frictionless counterpart.
  - Stability estimates used in the proof also allow to establish asymptotic expansions for small $|\gamma^1 - \gamma^2|$.
  - E.g., suppose $\beta^1 = -\beta^2 = \beta W_t$ and $\mathfrak{S} = bT + aW_T$, so that

    $$\bar{S}_t = (b - \bar{\gamma}a^2 T) + \bar{\gamma}a^2 t + aW_t$$

    Then price and volatility correction have opposite signs:

    $$S_0 \approx \bar{S}_0 + \frac{(\gamma^1-\gamma^2)\gamma^2}{\sqrt{2(\gamma^1+\gamma^2)}} T\beta a\sqrt{\lambda}$$
    $$\sigma_t \approx \bar{\sigma} - \frac{\gamma^1-\gamma^2}{\sqrt{2(\gamma^1+\gamma^2)}} \beta\sqrt{\lambda}$$

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Asymptotic Expansion ct'd

- Empirical literature going back to Amihud/Mendelson '86 consistently finds "illiquidity discounts".
- Necessarily correspond to a positive relationship between trading costs and volatility here.
  - Consistent with numerical evidence of Buss/Dumas '17, asymmetric information model of Danilova/Juillard '19 and empirics of Umlauf '93, Jones/Seguin '17, Hau '06.
- Corresponding expected returns approximately have Ornstein-Uhlenbeck dynamics.
  - Average level is higher for less liquid stocks – "liquidity premia" in line with empirical literature.
  - Partially predictable fluctuations like in reduced-form models in asset-management literature.
  - Not caused by mean-reverting fundamentals, but by sluggishness of the trading process.

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Calibrated Example

- Goal: choose model parameters to match main properties of time series data.
- Start with frictionless Bachelier model

$$S_t = (b - \bar{\gamma} s a^2 T) + \bar{\gamma} s a^2 t + a W_t$$

that obtains for $\beta^1 + \beta^2 = 0$ and $\mathfrak{S} = bT + aW_T$:

  - Match volatility a to standard deviation of asset returns.
  - Given supply s, choose aggregate risk aversion $\bar{\gamma}$ to match average asset returns.
  - Choose mean payoff b to match current asset price.
  - Individual risk aversions γ^1, γ^2 and endowment volatilities β^1, β^2 do not matter without frictions.

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Calibrated Example ct'd

- Without frictions: individual risk aversions γ^1, γ^2 and endowment volatilities β^1, β^2 do not matter.
- With frictions: frictionless equilibrium remains unchanged for $\gamma^1 = \gamma^2$. Size of deviation depends on $|\gamma^1 - \gamma^2|$.
- Gonon/M-K/Shi: fit to time series for prices *and* volume.
  - Choose quadratic trading costs to match effect of observable bid-ask spreads.
  - Choose endowment volatilities β^1, β^2 to match empirical "trading volume".
  - Choose heterogeneity $|\gamma^1 - \gamma^2|$ to match illiquidity discounts observed empirically.

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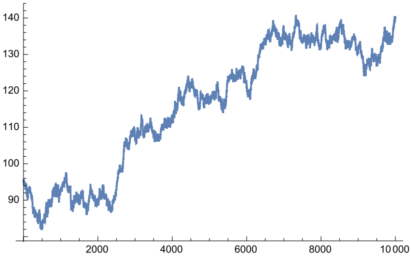
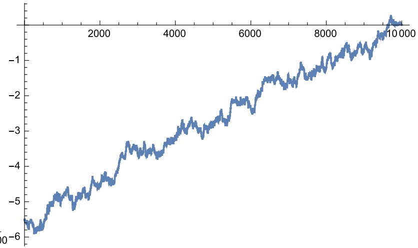
Calibrated Example ct'd

For $\gamma^2 = 2\gamma^1$:

- Frictionless equilibrium price is decreased far from maturity.
- Same terminal condition, and in turn higher expected returns ("liquidity premia").
- Frictionless equilibrium price and price adjustment:





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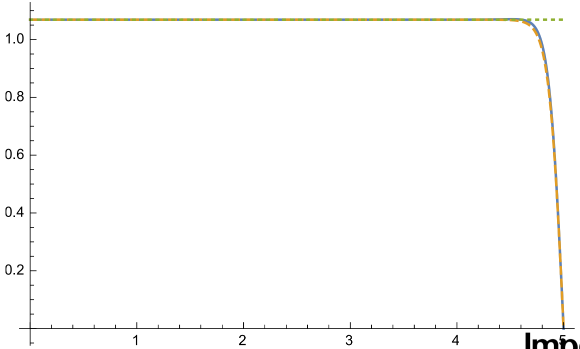
Calibrated Example ct'd

For $\gamma^2 = 2\gamma^1$:

- Frictionless equilibrium volatility is 15.
- Increase due to transaction costs of about 7%. Non negligible effect with realistic trading volume.



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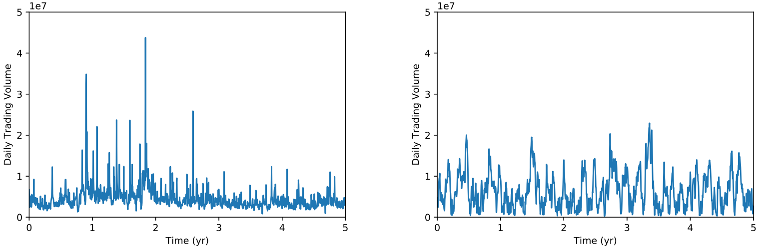
Calibrated Example ct'd

- Empirical trading volume (left panel) and simulated volume in the calibrated model (right panel).
- Level and diffusive behavior reproduced well. But excess skewness and kurtosis in data.



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Extensions

- Trades due to heterogenous beliefs rather than risk sharing?
  - Work in progress with Nutz and Tan.
- General small-cost expansions?
  - Work in progress with Shi and Weber.
- More general transaction costs.
  - E.g., proportional, square-root impact.
  - Leads to more complicated FBSDEs. Wellposedness unclear even for short horizons.
  - Gonon/Shi/MK: Numerical solution via deep-learning approach of Han/Jentzen/E '18.
  - Works well only for short time horizons so far.

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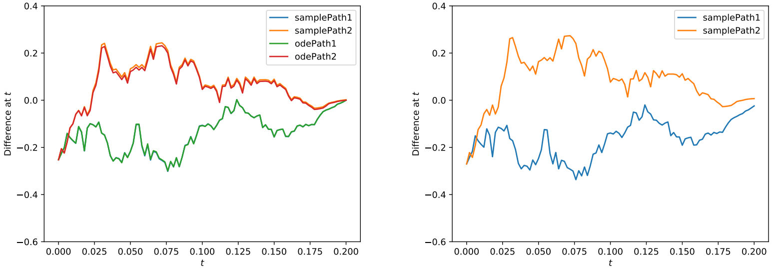
Extensions ct'd

Asset prices with trading costs $\propto \dot{\varphi}_t^2$ and $\propto \dot{\varphi}_t^{3/2}$:



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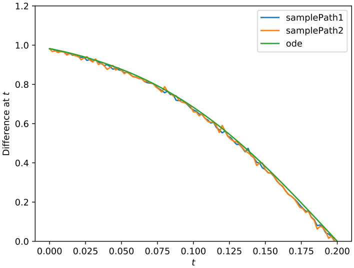
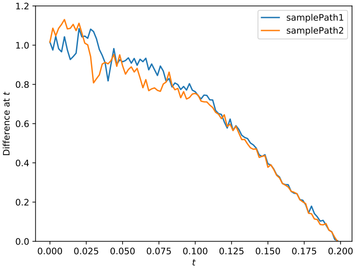
Extensions ct'd

Volatility corrections trading costs $\propto |\dot{\varphi}_t|^2$ and $\propto |\dot{\varphi}_t|^{3/2}$:





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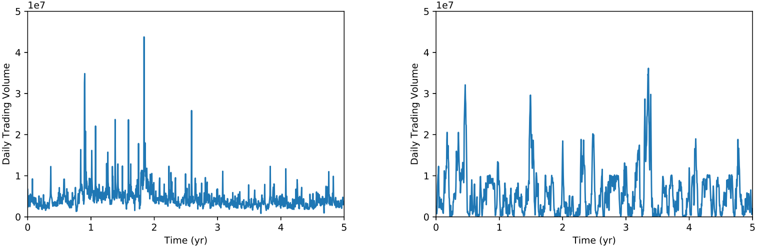
Extensions ct'd

- Empirical trading volume (left panel) and simulated volume in the calibrated model with costs $\propto |\dot{\varphi}_t|^{3/2}$ (right panel):



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